Generalized zeta functions, shape invariance and one-loop corrections to quantum Kink masses

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Abstract

We present a method to calculate the One-loop mass correction to Kinks mass in a $(1+1)$-dimensional field theoretical model in which the fluctuation potential $V''(\phi_c)$ has shape invariance property. We use the generalized zeta function regularization method to implement our setup for describing quantum kink states. PACS: 11.27.+d, 11.30.Pb, 03.65.Fd

1 Introduction

The quantum corrections to the mass of classical topological defects plays an important role in the semi-classical approach to the quantum field theory [1]. Computation of quantum energies around classical configurations in $(1+1)$-dimensional kinks has been developed in [2] by using topological boundary conditions, derivative expansion method [3], scattering phase shift technique [4], mode regularization approach [5], zeta-function regularization technique [6] and also dimensional regularization method [7]. In this paper we will give a derivation of the one-loop renormalized kink quantum mass correction in a $(1+1)$-dimensional scalar field theory model using generalized zeta function method for those potentials where the fluctuation potential, $V''(\phi_c)$ has the shape invariance property. This kind of potential is most occurrent in different fields of physics, particularly in quantum gravity and cosmology. What makes these potentials so important is that they posses a shape invariant operator in their prefactor, making corrections of these kind of potentials exact by the heat kernel method.

Consider the (1+1)-dimensional scalar field theory, where classical dynamics is described by following action functional for scalar $\phi(x)$ with potential $V(\phi)$

$$S[\phi] = \int d^2x \left[\frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right], \tag{1}$$

where a semicolon denotes ordinary derivation in two dimensional Minkowskian space-time. We shall consider the following examples

$$V(\phi) = \begin{cases} \frac{m^4}{\lambda}\left(1-\cos(\frac{\sqrt{\lambda}\phi}{m})\right); & Sine-Gordon(SG)\ model \\ \\ \frac{m^4}{4\lambda}\left((\frac{\sqrt{\lambda}}{m}\phi)^2-1\right)^2; & \phi^4-model. \end{cases} \tag{2}$$

These two kinds of potentials are tex-book cases. The method that we use can easily be developed to include all potentials that have shape invariant property. There are a number of papers where this kind of potentials are discussed, see for example [8]. We know that mass m and coupling constant λ may be scaled out so that

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semi-classical expansion is an expansion in powers of $\frac{\hbar\lambda}{m^2}$ and therefore we can set $m = \lambda = 1$. Classical static kink-antikink solutions $\phi_c(x)$ satisfy

$$\phi_c' = \pm\sqrt{2V(\phi_c)}. \tag{3}$$

Therefore for potentials in (2), the classical solutions are

$$\phi_c = \begin{cases} 4\arctan(e^x); & SG\ model \\ \\ \tanh(\sqrt{2}x); & \phi^4 - model. \end{cases} \tag{4}$$

As we will see in the next section, to compute mass corrections we need the fluctuation potential $V''(\phi_c(x))$. For cases listed above, we have

$$V''(\phi_c) = \begin{cases} 1 - \frac{2}{\cosh^2 x}; & SG\ model \\ \\ 4 - \frac{6}{\cosh^2 x}; & \phi^4 - model. \end{cases} \tag{5}$$

In the next section we quote briefly the generalized zeta function method. We then use shape invariance property of potentials (5) and the heat kernel method to obtain quantum corrections to kink masses.

2 Semi-Classical Quantum Kink States

Classical configuration space is found by static configuration $\Phi(x)$, so that the energy functional corresponding to classical action functional (1)

$$E[\Phi] = \int dx \left[\frac{1}{2}\Phi_{,\mu}\Phi^{,\mu} + V(\Phi)\right], \tag{6}$$

is finite. One can describe quantum evolution in Schrodinger picture by the following functional equation

$$i\hbar\frac{\partial}{\partial t}\Phi[\phi(x),t] = H\Phi[\phi(x),t], \tag{7}$$

so that quantum Hamiltonian operator is given by

$$H = \int dx \left[-\frac{\hbar^2}{2}\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x)} + E[\phi]\right]. \tag{8}$$

In field representation matrix elements of evolution operator are given by

$$G(\phi^{(f)}(x),\phi^{(i)}(x),T) = \langle\phi^{(f)}|e^{-\frac{iT}{\hbar}H}|\phi^{(i)}\rangle = \int D[\phi(x,t)]\exp\left(\frac{-i}{\hbar}S[\phi]\right), \tag{9}$$

where the initial conditions are those of static solutions kink of classical equations where $\phi^{(i)}(x,0) = \phi_k(x)$, $\phi^{(f)}(x,T) = \phi_k(x)$. In semi-classical picture, we are interested in loop expansion for evolution operator up to the first quantum correction

$$G(\phi^{(f)}(x),\phi^{(i)}(x),\beta) = \exp\left(-\frac{\beta}{\hbar}E[\phi_k]\right)Det^{-\frac{1}{2}}\left[-\partial_\tau^2 + P\Delta\right](1+\mathcal{O}(\hbar)), \tag{10}$$

where we use analytic continuation to Euclidean time, $t = -i\tau$,$T = -i\beta$, and Δ is the differential operator

$$\Delta = -\frac{d^2}{dx^2} + \frac{d^2V}{d\phi^2}\ |_{\phi=\phi_k}, \tag{11}$$

P is the projector over the strictly positive part of spectrum of Δ

$$\Delta\xi_n(x) = \omega_n^2\xi_n(x), \quad \omega_n^2\ \epsilon\ Spec(\Delta) = Spec(P\Delta) + \{0\}. \tag{12}$$

We write functional determinant in the form

$$Det\left[-\frac{\partial^2}{\partial\tau^2} + \Delta\right] = \prod_n det\left[-\frac{\partial^2}{\partial\tau^2} + \omega_n^2\right]. \tag{13}$$

All determinants in infinite product correspond to harmonic oscillators of frequency ω_n. On the other hand, it is well known that [10]

$$\begin{aligned} det\left(-\frac{\partial^2}{\partial\tau^2}+\omega_n^2\right)^{-\frac{1}{2}} &= \prod_{j=1}^{N}\left(\frac{j^2\pi^2}{\beta^2}+\omega_n^2\right)^{-\frac{1}{2}} \\ &= \prod_j\left(\frac{j^2\pi^2}{\beta^2}\right)^{-\frac{1}{2}}\prod_j\left(1+\frac{\omega_n^2\beta^2}{j^2\pi^2}\right)^{-\frac{1}{2}}. \end{aligned} \tag{14}$$

The first product dose not depend on ω_n and combines with the Jacobian and other factors we have collected into a single constant. The second factor has the limit $\left[\frac{\sinh(\omega_n\beta)}{\omega_n\beta}\right]^{-\frac{1}{2}}$, and thus, with an appropriate normalization, we obtain for large β

$$G(\phi^{(f)}(x),\phi^{(i)}(x),\beta) \cong \exp\left(-\frac{\beta}{\hbar}E[\phi_k]\right)\prod_n\left(\frac{\omega_n}{\pi\hbar}\right)^{\frac{1}{2}}\exp\left(-\frac{\beta}{2}\sum_n\omega_n(1+\mathcal{O}(\hbar))\right) \tag{15}$$

where eigenvalues in the kernel of Δ have been excluded. Interesting eigenenergy wave functionals

$$H\Phi_j[\phi_k(x)] = \varepsilon_j\Phi_j[\phi_k(x)] \tag{16}$$

we have an alternative expression for G_E for $\beta\to\infty$.

$$G(\phi^{(f)}(x),\phi^{(i)}(x),\beta) \cong \Phi_0^*[\phi_k(x)]\Phi_0[\phi_k(x)]\exp\left(-\beta\frac{\varepsilon_0}{\hbar}\right), \tag{17}$$

and, therefore, from (14) and (16) we obtain

$$\varepsilon_0 = E[\phi_k]+\frac{\hbar}{2}\sum_{\omega_n^2>0}\omega_n+\mathcal{O}(\hbar), \tag{18}$$

$$|\Phi_0[\phi_k(x)]|^2 = Det^{\frac{1}{4}}\left[\frac{P\Delta}{\pi^2\hbar^2}\right], \tag{19}$$

as the Kink ground state energy and wave functional up to One-Loop order.
If we define the generalized zeta function

$$\zeta_{P\triangle} = Tr(P\Delta)^{-s} = \sum_{\omega_n^2>0}\frac{1}{(\omega_n^2)^s}, \tag{20}$$

associated to differential operator $P\triangle$, then

$$\varepsilon_0^k = E[\phi_k]+\frac{\hbar}{2}Tr(P\Delta)^{\frac{1}{2}}+\mathcal{O}(\hbar^2) = E[\phi_k]+\frac{\hbar}{2}\zeta_{P\Delta}\left(-\frac{1}{2}\right)+\mathcal{O}(\hbar^2). \tag{21}$$

The eigenfunction of Δ is a basis for quantum fluctuations around kink background, therefore sum of the associated zero-point energies encoded in $\zeta_{P\Delta}(-\frac{1}{2})$ in (20) is infinite. According to zeta function regularization procedure, energy and mass renormalization prescription, renormalized kink energy in semi-classical limit becomes [9]

$$\varepsilon^k(s) = E[\phi_k]+\Delta M_k+\mathcal{O}(\hbar^2) = E[\phi_k]+\lim_{s\to\frac{-1}{2}}[\delta_1\varepsilon^k(s)+\delta_2^k\varepsilon(s)]+\mathcal{O}(\hbar^2), \tag{22}$$

where

$$\begin{cases} \delta_1\varepsilon^k(s) = \frac{\hbar}{2}\mu^{2s+1}[\zeta_{P\Delta}(s)-\zeta_\nu(s)], \\ \delta_2\varepsilon^k(s) = \lim_{L\to\infty}\frac{\hbar}{2L}\mu^{2s+1}\frac{\Gamma(s+1)}{\Gamma(s)}\zeta_\nu(s+1)\int_{-\frac{L}{2}}^{\frac{L}{2}}dx\left[\frac{d^2V}{d\phi^2}|_{\phi_k}-\frac{d^2V}{d\phi^2}|_{\phi_\nu}\right]. \end{cases} \tag{23}$$

Here ϕ_ν is a constant minimum of potential $V(\phi)$, E is corresponding classical energy where μ has the unit $length^{-1}$ dimension, introduced to make the terms in (23) homogeneous from a dimensional point of view and

ζ_ν denoted zeta function associated with vacuum ϕ_v.
Now we explain very briefly how one can calculate zeta function of an operator though heat kernel method. We introduce generalized Riemann zeta function of operator A by

$$\zeta_A(s) = \sum_n \frac{1}{|\lambda_n|^s}, \tag{24}$$

where λ_n are eigenvalues of operator A. On the other hand, $\zeta_A(s)$ is the Mellin transformation of heat kernel $G(x, y, t)$ which satisfies the following heat diffusion equation

$$AG(x, y, t) = -\frac{\partial}{\partial t} G(x, y, t), \tag{25}$$

with an initial condition $G(x, y, 0) = \delta(x - y)$. Note that $G(x, y, t)$ can be written in terms of its spectrum

$$G(x, y, t) = \sum_n e^{-\lambda_n t} \psi_n^*(x) \psi_n(y), \tag{26}$$

and as usual, if the spectrum is continues, one should integrate it. From relation (17), it is clear that

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \int_{-\infty}^\infty G(x, x, \tau) dx. \tag{27}$$

Hence, if we know the associated Green function of an operator, we can calculate generalized zeta function corresponding to that operator. In the next sections we calculate the Green function of ϕ^4-model and SG via shape invariance property and there, by using equations (25), (26) and (27) we will obtain one-loop corrections to quantum kink masses.

3 Quantum Mass of SG and ϕ^4-models

In this section we calculate one-loop quantum mass of these two potentials. According to the previous section the second derivative of these potentials at the Kink solution can be written as

$$U(x) = l^2 - \frac{l(l+1)}{cosh^2(x)}, \tag{28}$$

so that for $l = 1$ and $l = 2$ we obtain SG and ϕ^4-model second derivative potentials respectively. Therefor the operator (11) which acts on the eigenfunctions becomes

$$\Delta_l = -\frac{d^2}{dx^2} + l^2 - \frac{l(l+1)}{cosh^2(x)}. \tag{29}$$

Also the operator acting on the vacuum has the following form

$$\Delta_l(0) = -\frac{d^2}{dx^2} + l^2. \tag{30}$$

In the reminding of this section, to obtaining the spectrum of (29) we will use the shape invariance property. First we review briefly concepts that we will use.
Consider the following one-dimensional bound-state Hamiltonian

$$H = -\frac{d^2}{dx^2} + U(x), \qquad\qquad x \in I \subset \mathbb{R} \tag{31}$$

where I is the domain of x and $U(x)$ is a real function of x, which can be singular only in the boundary points of the domain. Let us denote by E_n and $\psi_n(x)$ the eigenvalues and eigenfunctions of H respectively. We use factorization method which consists of writing Hamiltonian as the product of two first order mutually adjoint differential operators A and $A^\dagger$. If the ground state eigenvalue and eigenfunctions are known, then one can factorize Hamiltonian (31) as

$$H = A^\dagger A + E_0, \tag{32}$$

where E_0 denotes the ground-state eigenvalue,

$$\begin{aligned} A &= \tfrac{d}{dx} + W(x), \\ A^\dagger &= -\tfrac{d}{dx} + W(x), \end{aligned} \tag{33}$$

and

$$W(x) = -\frac{d}{dx}\ln(\psi_0). \tag{34}$$

Supersymmetric quantum mechanics (SUSY QM) begins with a set of two matrix operators, known as supercharges

$$Q^+ = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \qquad\qquad Q^- = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}. \tag{35}$$

This operators form the following superalgebra [11]

$$\{Q^+, Q^-\} = H_{SS}, \qquad [H_{SS}, Q^\pm] = (Q^\pm)^2 = 0, \tag{36}$$

where SUSY Hamiltonian H_{SS} is defined as

$$H_{SS} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{37}$$

In terms of the Hamiltonian supercharges

$$\begin{aligned} Q_1 &= \tfrac{1}{\sqrt{2}}(Q^+ + Q^-), \\ Q_2 &= \tfrac{1}{\sqrt{2}i}(Q^+ - Q^-), \end{aligned} \tag{38}$$

the superalgebra takes the form

$$\{Q_i, Q_j\} = H_{SS}\delta_{ij}, \qquad [H_{SS}, Q_i] = 0, \qquad i, j = 1, 2. \tag{39}$$

The operators H_1 and H_2

$$\begin{aligned} H_1 &= A^\dagger A = -\tfrac{d^2}{dx^2} + U_1 = -\tfrac{d^2}{dx^2} + W^2 - \tfrac{dW}{dx}, \\ H_2 &= AA^\dagger = -\tfrac{d^2}{dx^2} + U_2 = -\tfrac{d^2}{dx^2} + W^2 + \tfrac{dW}{dx}, \end{aligned} \tag{40}$$

are called SUSY partner Hamiltonians and the function W is called the superpotential. Now, let us denote by $\psi_l^{(1)}$ and $\psi_l^{(2)}$ the eigenfunctions of H_1 and H_2 with eigenvalues $E_l^{(1)}$ and $E_l^{(2)}$, respectively. It is easy to see that the eigenvalues of the above Hamiltonians are positive and isospectral, i.e., they have almost the same energy eigenvalues, except for the ground state energy of H_1. According to the [11], their energy spectra are related as

$$\begin{aligned} E_l &= E_l^{(1)} + E_0, \qquad E_0^{(1)} = 0, \quad \psi_l = \psi_l^{(1)}, \quad l = 0, 1, 2, .., \\ E_l^{(2)} &= E_{l+1}^{(1)}, \\ \psi_l^{(2)} &= [E_{l+1}^{(1)}]^{-\frac{1}{2}} A\psi_{l+1}^{(1)}, \\ \psi_{l+1}^{(1)} &= [E_l^{(2)}]^{-\frac{1}{2}} A^\dagger \psi_l^{(2)}. \end{aligned} \tag{41}$$

Therefor if the eigenvalues and eigenfunctions of H_1 were known, one could immediately derive the spectrum of H_2. However the above relations only give the relationship between the eigenvalues and eigenfunctions of the two partner Hamiltonians. A condition of an exactly solvability is known as the shape invariance condition. This condition means the pair of SUSY partner potentials $U_{1,2}(x)$ are similar in shape and differ only in the parameters that appears in them [12],

$$U_2(x; a_1) = U_2(x; a_2) + \mathcal{R}(a_1), \tag{42}$$

where a_1 is a set of parameters and a_2 is a function of a_1. Then the eigenvalues of H_1 are given by

$$E_l^{(1)} = \mathcal{R}(a_1) + \mathcal{R}(a_2) + ... + \mathcal{R}(a_l), \tag{43}$$

and the corresponding eigenfunctions are

$$\psi_l = \prod_{m=1}^{l} \frac{A^\dagger(x; a_m)}{\sqrt{E_m}} \psi_0(x; a_{l+1}). \tag{44}$$

The shape invariance condition (42) can be rewritten in terms of the factorization operators defined in equation (33)

$$A(x; a_1)A^\dagger(x; a_1) = A^\dagger(x; a_2)A(x; a_2) + \mathcal{R}(a_1), \tag{45}$$

where $a_2 = f(a_1)$.
Now we are ready to obtain spectra of Δ_l operator defined in (29). For a given eigenspectrum of E_l, we introduce the following factorization operators

$$\begin{aligned} A_l &= \tfrac{d}{dx} + l \tanh(x), \\ A_l^\dagger &= -\tfrac{d}{dx} + l \tanh(x), \end{aligned} \tag{46}$$

the operator Δ_l can be factorized as

$$\begin{aligned} A_l^\dagger(x)A_l(x)\psi_n^{(1)}(x) &= E_n^{(1)}\psi_n^{(1)}(x), \\ A_l(x)A_l^\dagger(x)\psi_n^{(2)}(x) &= E_n^{(2)}\psi_n^{(2)}(x). \end{aligned} \tag{47}$$

Therefor for a given l, its first bounded excited state can be obtained from the ground state of $l-1$ and consequently the excited state m of a given l, $\psi_{l,m}(x)$, using (44) can be written as

$$\psi_{l,m}(x) = \sqrt{\frac{2(2m-1)!}{\Pi_{j=1}^m j(2l-j)}} \frac{1}{2^m(m-1)!} A_l^\dagger(x)A_{l-1}^\dagger(x)...A_{m+1}^\dagger(x)\frac{1}{\cosh^m(x)}, \tag{48}$$

with eigenvalue $E_{l,m} = m(2l-m)$. Obviously its ground state with $E_{l,0} = 0$ is given by $\psi_{l,0} \propto \cosh^{-l}(x)$. Also its continuous spectrum consists of

$$\psi_{l,k}(x) = \frac{A_l^\dagger(x)}{\sqrt{k^2+l^2}} \frac{A_{l-1}^\dagger(x)}{\sqrt{k^2+(l-1)^2}} ... \frac{A_1^\dagger(x)}{\sqrt{k^2+1}} \frac{e^{ikx}}{\sqrt{2\pi}}, \tag{49}$$

with eigenvalues $E_{l,k} = l^2 + k^2$ with following normalization condition

$$\int_{-\infty}^{\infty} \psi_{l,k}^*(x)\psi_{l,k'}(x)dx = \delta(k-k'). \tag{50}$$

Therefor, using equations (25), (26), (48) and (49) we find

$$G_{\Delta_l(0)}(x, y, \tau) = \frac{e^{-l^2\tau}}{2\sqrt{\pi\tau}} e^{-(x-y)^2/4\tau}, \tag{51}$$

and

$$\begin{aligned} G_{\Delta_l}(x, y, \tau) &= \textstyle\sum_{m=1}^{l-1} \psi_{l,m}^*(x)\psi_{l,m}(y)e^{-m(2l-m)\tau} \\ &+ \textstyle\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-(l^2+k^2)\tau}}{\prod_{m=1}^l (k^2+m^2)} \left(\prod_{m=1}^l A_m^\dagger(x)e^{ikx}\right)^* \left(\prod_{m=1}^l A_m^\dagger(y)e^{iky}\right). \end{aligned} \tag{52}$$

Hence, for $l = 1$ (SG), according to (27) it is easy to show that

$$\xi_{P\Delta_1}(s) - \xi_{\Delta_1(0)}(s) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dk}{(k^2+1)^{s+1}} = -\frac{1}{\sqrt{\pi}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}. \tag{53}$$

Consequently according to the (23) the first correction term to the kink quantum mass of SG becomes

$$\delta_1\varepsilon^k(s) = \frac{\hbar}{2}\mu^{2s+1}\left[\xi_{P\Delta_1}(s) - \xi_{\Delta_1(0)}(s)\right] = -\frac{\hbar}{2\sqrt{\pi}}\mu^{2s+1}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}. \tag{54}$$

The second correction term is also given by

$$\begin{aligned}\delta_2\varepsilon^k(s) &= \lim_{L\to\infty}\tfrac{\hbar}{2L}\mu^{2s+1}\tfrac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}\xi_{\Delta_1(0)}(s+1)\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(1-\tfrac{2}{\cosh^2(x)}-1\right)dx\\ &= -\lim_{L\to\infty}\tfrac{\hbar}{2L}\mu^{2s+1}\tfrac{\Gamma(s+1)}{\Gamma(s)}\tfrac{L}{2\sqrt{\pi}\Gamma(s+1)}\Gamma(s+\tfrac{1}{2})2\tanh(\tfrac{L}{2}) = \\ &\quad -\tfrac{\hbar}{\sqrt{\pi}}\mu^{2s+1}\tfrac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}\end{aligned} \tag{55}$$

Therefore the corrected mass for SG kink is derived from

$$\varepsilon^k(s) = E[\phi_k] + \lim_{s\to-\frac{1}{2}}\left[\delta_1\varepsilon^k(s) + \delta_2\varepsilon^k(s)\right], \tag{56}$$

Using the variable $\alpha = s+\frac{1}{2}$, functions $\delta_1\varepsilon^k(s)$ and $\delta_2\varepsilon^k(s)$ can be written in the following form

$$\begin{cases}\delta_1\varepsilon^k(\alpha) = -\tfrac{\hbar}{2\sqrt{\pi}}\mu^{2s}\tfrac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2})},\\ \delta_1\varepsilon^k(\alpha) = -\tfrac{\hbar}{\sqrt{\pi}}\mu^{2s}\tfrac{\Gamma(\alpha)}{\Gamma(\alpha-\frac{1}{2})},\end{cases} \tag{57}$$

Now by using the Gamma function properties, we have

$$\begin{cases}\delta_1\varepsilon^k(0) = -\tfrac{\hbar}{2\sqrt{\pi}}\lim_{\alpha\to 0}\mu^{2\alpha}\left[\tfrac{1}{\sqrt{\pi}\alpha} - \tfrac{\gamma-\Psi(\frac{1}{2})}{\sqrt{\pi}} + \mathcal{O}(\alpha)\right],\\ \delta_1\varepsilon^k(\alpha) = -\tfrac{\hbar}{\sqrt{\pi}}\lim_{\alpha\to 0}\mu^{2\alpha}\left[\tfrac{1}{2\sqrt{\pi}\alpha} + \tfrac{\gamma+\Psi(-\frac{1}{2})}{2\sqrt{\pi}} + \mathcal{O}(\alpha)\right],\end{cases} \tag{58}$$

where $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is digamma function and γ is the Euler-Mascheroni constant . Sum of contributions of two poles leaves a finite remainder and we end with the finite answer

$$\delta_1\varepsilon^k + \delta_2\varepsilon^k = -\frac{m\hbar}{\pi} \quad , \quad \varepsilon^k = E[\phi_k] - \frac{m\hbar}{\pi} + \mathcal{O}(\hbar^2\gamma). \tag{59}$$

$$E[\phi_k] = \frac{8m}{\gamma}$$

$$\varepsilon^k = \frac{8m}{\gamma} - \frac{m\hbar}{\pi} + \mathcal{O}(\hbar^2\gamma). \tag{60}$$

The one-loop correction to SG kink obtained by means of generalized zeta function procedure exactly agrees with accepted result, see [10], [11], [12], [13] and henceforth, with outcome of the mode number regularization method, [14]. In the case of ϕ^4-model we left with $l=2$ and then using (27) we have

$$\int_{-\infty}^{\infty}\left[G_{\nabla_2}(x,x,\tau) - G_{\nabla_2(0)}(x,x,\tau)\right]dx = e^{-3\tau} - \frac{3}{\pi}e^{-4\tau}\int_{-\infty}^{\infty}\frac{(k^2+2)e^{-k^2\tau}}{(k^2+1)(k^2+4)}, \tag{61}$$

and using (27) we obtain

$$\begin{aligned}\xi_{\nabla_2}(s) - \xi_{\nabla_2(0)}(s) &= 3^{-s} - \tfrac{3}{\pi}\int_{-\infty}^{\infty}\tfrac{dk}{(k^2+4)^{s+1}} - \tfrac{3}{\pi}\int_{-\infty}^{\infty}\tfrac{dk}{(k^2+1)(k^2+4)^{s+1}} = \\ &3^{-s} - \tfrac{3}{\sqrt{\pi}}2^{-(2s+1)}\tfrac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} - \tfrac{3}{\sqrt{\pi}}2^{-(2s+3)}\tfrac{\Gamma(s+\frac{3}{2})}{\Gamma(s+2)}{}_2F_1[s+\tfrac{3}{2},1,s+2,\tfrac{3}{4}],\end{aligned} \tag{62}$$

where we have used the well-known Feynman integral

$$\begin{aligned}&\tfrac{1}{D_1^{a_1}D_2^{a_2}\ldots D_n^{a_n}} = \\ &\tfrac{\Gamma(a_1+a_2+\ldots+a_n)}{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_n)}\int dt_1dt_2\ldots dt_n\tfrac{\delta(1-t_1-t_2-\ldots-t_n)t_1^{a_1-1}\ldots t_n^{a_n-1}}{(t_1D_1+\ldots+t_nD_n)^{a_1+\cdots+a_n}}.\end{aligned} \tag{63}$$

Consequently we have

$$\delta_1\varepsilon^k(s)=\tfrac{\hbar}{2}\mu^{2s+1}\left[\xi_{P\Delta_2}(s)-\xi_{\Delta_0}(s)\right]=$$
$$\tfrac{\hbar}{2}\left(3^{-s}-\tfrac{3}{\sqrt{\pi}}2^{-(2s+1)}\tfrac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}-\tfrac{3}{\sqrt{\pi}}2^{-(2s+3)}\tfrac{\Gamma(s+\frac{3}{2})}{\Gamma(s+2)}{}_2F_1[s+\tfrac{3}{2},1,s+2,\tfrac{3}{4}]\right). \tag{64}$$

Also we obtain

$$\delta_2\varepsilon^k(s)=\lim_{L\to\infty}\tfrac{\hbar}{2L}\mu^{2s+1}\tfrac{\Gamma(s+1)}{\Gamma(s)}\xi_{\Delta_2(0)}(s+1)\int_{-\frac{L}{2}}^{\frac{L}{2}}dx(-6\cosh^{-2}(x))$$
$$=-\tfrac{3\hbar}{\sqrt{\pi}2^{2s+1}}\mu^{2s+1}\tfrac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}. \tag{65}$$

Finally we have

$$\lim_{s\to-\frac{1}{2}}(\delta_1\varepsilon^k(s)+\delta_2\varepsilon^k(s))=\tfrac{\sqrt{3}}{2}\hbar-\tfrac{3\hbar}{8\sqrt{\pi}}\tfrac{\Gamma(1)}{\Gamma(\frac{3}{2})}{}_2F_1[1,1,\tfrac{3}{2},\tfrac{3}{4}]$$
$$-\lim_{\alpha\to0}\left(\tfrac{3\hbar}{\sqrt{\pi}}\tfrac{\Gamma(\alpha)}{\Gamma(\alpha-\frac{1}{2})}+\tfrac{3\hbar}{2\sqrt{\pi}}\tfrac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2})}\right) \tag{66}$$
$$=\tfrac{\hbar}{2\sqrt{3}}-\tfrac{3\hbar}{\pi}.$$

Now using $E[\phi_k]=4m^3/3\lambda$, we find

$$\varepsilon^k=\frac{4m^3}{3\lambda}+m\hbar\left(\frac{1}{2\sqrt{3}}-\frac{3}{\pi}\right), \tag{67}$$

the same answer offered by mode-number regularization method in [15].

4 Conclusion

In this article we used the shape invariance property of fluctuation operator of SG and ϕ^4-models to obtain one-loop quantum correction to the kink mass. This method can be extend to those quantum fields that their fluctuation operators have shape invariance property. An interesting extension worth studying is to use this method for quantum fields in $(1+2)$-dimension.

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